

Universal Algorithm for Online Trading Based on the Method of Calibration

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Abstract

We present a universal algorithm for online trading in Stock Market which performs asymptotically at least as good as any stationary trading strategy that computes the investment at each step using a fixed function of the side information that belongs to a given RKHS (Reproducing Kernel Hilbert Space). Using a universal kernel, we extend this result for any continuous stationary strategy. In this learning process, a trader rationally chooses his gambles using predictions made by a randomized well-calibrated algorithm. Our strategy is based on Dawid's notion of calibration with more general checking rules and on some modification of Kakade and Foster's randomized rounding algorithm for computing the well-calibrated forecasts. We combine the method of randomized calibration with Vovk's method of defensive forecasting in RKHS. Unlike the statistical theory, no stochastic assumptions are made about the stock prices. Our empirical results on historical markets provide strong evidence that this type of technical trading can "beat the market" if transaction costs are ignored.

Keywords: asymptotic calibration, defensive forecasting, online trading, reproducing kernel Hilbert space, universal kernel, universal trading strategy, stationary trading strategy with a side information

1. Introduction

Predicting sequences is the key problem for machine learning, computational finance and statistics. These predictions can serve as a base for developing the efficient methods for playing financial games in Stock Market.

The learning process proceeds as follows: observing a finite-state sequence given online, a forecaster assigns a subjective estimate to future states.

A minimal requirement for testing any prediction algorithm is that it should be calibrated (cf. Dawid 1982). Dawid gave an informal explanation of calibration for binary outcomes. Let a sequence $\omega_1, \omega_2, \dots, \omega_{n-1}$ of binary outcomes be observed by a forecaster

whose task is to give a probability p_n of a future event $\omega_n = 1$. In a typical example, p_n is interpreted as a probability that it will rain. Forecaster is said to be well-calibrated if it rains as often as he leads us to expect. It should rain about 80% of the days for which $p_n = 0.8$, and so on.

A more precise definition is as follows. Let $I(p)$ denote the characteristic function of a subinterval $I \subseteq [0, 1]$, i.e., $I(p) = 1$ if $p \in I$ and $I(p) = 0$, otherwise. An infinite sequence of forecasts p_1, p_2, \dots is calibrated for an infinite binary sequence of outcomes $\omega_1 \omega_2 \dots$ if for characteristic function $I(p)$ of any subinterval of $[0, 1]$ the calibration error tends to zero, i.e.,

$$\frac{1}{n} \sum_{i=1}^n I(p_i)(\omega_i - p_i) \rightarrow 0$$

as $n \rightarrow \infty$. The indicator function $I(p_i)$ determines some “checking rule” that selects indices i , where we compute the deviation between forecasts p_i and outcomes ω_i .

If the weather acts adversatively, then, as shown by Oakes (1985) and Dawid (1985), any deterministic forecasting algorithm will not always be calibrated.

Foster and Vohra (1998) show that calibration is almost surely guaranteed with a randomizing forecasting rule, i.e., where the forecasts p_i are chosen using internal randomization and the forecasts are hidden from the weather until the weather makes its decision whether to rain or not.

The origin of the calibration algorithms is the Blackwell (1956) approachability theorem but, as its drawback, the forecaster has to use linear programming to compute the forecasts. We modify and generalize a more computationally efficient method from Kakade and Foster (2004), where “an almost deterministic” randomized rounding universal forecasting algorithm is presented. For any sequence of outcomes $\omega_1, \omega_2, \dots$ and for any precision of rounding $\Delta > 0$, an observer can simply randomly round the deterministic forecast p_i up to Δ to a random forecast \tilde{p}_i in order to calibrate for this sequence with probability one:

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n I(\tilde{p}_i)(\omega_i - \tilde{p}_i) \right| \leq \Delta, \quad (1)$$

where $I(p)$ is the characteristic function of any subinterval of $[0, 1]$. This algorithm can be easily generalized such that the calibration error tends to zero as $n \rightarrow \infty$.

Kakade and Foster and others considered a finite outcome space and a probability distribution as the forecast. In this paper, the outcomes ω_i are real numbers from unit interval $[0, 1]$ and the forecast p_i is a single real number (which can be an output of a random variable). This setting is closely related to Vovk (2005a) defensive forecasting approach (see below).

In this case real valued predictions $p_i \in [0, 1]$ could be interpreted as mean values of future outcomes under some unknown to us probability distributions in $[0, 1]$. We do not know precise form of such distributions – we should predict only future means.

The well known applications of the method of calibration belong to different fields of the game theory and machine learning. Kakade and Foster proved that empirical frequencies of play in any normal-form game with finite strategy sets converges to a set of correlated equilibrium if each player chooses his gamble as the best response to the well calibrated forecasts

of the gambles of other players. In series of papers: Vovk et al. (2005), Vovk (2005a), Vovk (2006), Vovk (2006a), Vovk (2007), Vovk developed the method of calibration for the case of more general RKHS and Banach spaces. Vovk called his method defensive forecasting (DF). He also applied his method for recovering unknown functional dependencies presented by arbitrary functions from RKHS and Banach spaces. Chernov et al. (2010) show that well-calibrated forecasts can be used to compute predictions for the Vovk (1997) aggregating algorithm. In defensive forecasting, continuous loss (gain) functions are considered.

In this paper we present a new application of the method of calibration. We construct “a universal” strategy for online trading in *Stock Market* which performs asymptotically at least as good as any not “too complex” trading strategy D . Technically, we are interested in the case where the trading strategy D is assumed to belong to a large reproducing kernel Hilbert space (to be defined shortly) and the complexity of D is measured by its norm. Using a universal kernel, we extend this result to any continuous stationary trading strategy. Our universal trading strategy is represented by a discontinuous function though it uses a randomization.

The problem of construction the universal strategies for online trading in Stock Market is popular in Machine Learning. The worst case study of universal trading was introduced by Cover (1991). Unlike many authors, we consider the simplest case: online trading with only stock. These results can be generalized for the case of several stocks and for dynamical portfolio hedging in sense of framework proposed by Cover and Ordentlich (1996).

We consider a game with players: *Stock Market* and *Trader*. At the beginning of each round i *Trader* is shown an object \mathbf{x}_i which contains a side information. Past prices of the stock S_1, \dots, S_{i-1} are also given for *Trader* (they can be considered as a part of the side information). Using this information, *Trader* announces a number M_i of shares of the stock he wants to purchase by S_{i-1} each. At the end of the round i *Stock Market* announces the price S_i of the stock, and *Trader* receives his gain or suffers loss $M_i(S_i - S_{i-1})$ for round i . The total gain or loss for the first n rounds is equal to $\sum_{i=1}^n M_i(S_i - S_{i-1})$.

We show that, using the well-calibrated forecasts, it is possible to construct a universal strategy for online trading in the Stock Market which performs asymptotically at least as good as any stationary trading strategy presented by a continuous function D from the object \mathbf{x}_i . This universal trading strategy is of decision type: we buy or sell only one share of the stock at each round. The learning process is the most traditional one. At each step, *Trader* makes a randomized prediction \tilde{p}_i of a future price S_i of the stock and takes the best response to this prediction. He chooses a strategy: to dealing for a rise: $\tilde{M}_i = 1$ if $\tilde{p}_i > \tilde{S}_{i-1}$, or to dealing for a fall: $\tilde{M}_i = -1$ otherwise, where \tilde{S}_{i-1} is the randomized past price of the stock. *Trader* uses some randomized algorithm for computing the well-calibrated forecasts \tilde{p}_i .

Therefore, our universal strategy uses some internal randomization. Unlike the statistical theory, no stochastic assumptions are made about the evolution of stock prices.

Our main result, Theorems 4 and 5 (Section 4), and Theorem 7 (Section 5), says that this trading strategy \tilde{M}_i performs asymptotically at least as good as any stationary trading strategy presented by a continuous function $D(x)$. With probability one, the gain of this trading strategy is asymptotically not less than the average gain of any stationary trading

strategy from one share of the stock:

$$\liminf_{n \rightarrow \infty} \left(\sum_{i=1}^n \tilde{M}_i(S_i - S_{i-1}) - \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - S_{i-1}) \right) \geq 0,$$

where \mathbf{x}_i is a side information used by the stationary trading strategy D at step i and $\|D\|_\infty = \sup_x |D(x)|$.

To achieve this goal we extend in Theorem 1 (Section 3) Kakade and Foster's forecasting algorithm for a case of arbitrary real valued outcomes and to a more general notion of calibration with changing parameterized checking rules. We combine it with Vovk et al. (2005) defensive forecasting method in RKHS (cf. Vovk 2005a). In Section 5, using a universal kernel, we generalize this result to any continuous stationary trading strategy. We show in Section 6 that the universality property fails if we consider discontinuous trading strategies. On the other hand, we show in Theorem 9 that a universal trading strategy exists for a class of randomized discontinuous trading strategies.

In Section 7 results of numerical experiments are presented. Our empirical results on historical markets provide strong evidence that this type of online trading can beat the market: our universal strategy is always better than "buy-and-hold" strategy for each stock chosen arbitrarily in Stock Market. This strategy outperforms also an online trading strategy using some standard prediction algorithm (ARMA).

2. Preliminaries

By a kernel function on a set X we mean any function $K(x, y)$ which can be represented as a dot product $K(x, y) = (\Phi(x) \cdot \Phi(y))$, where Φ is a mapping from X to some Hilbert feature space.

The reproducing kernels are of special interest. A Hilbert space \mathcal{F} of real-valued functions on a compact metric space X is called RKHS (Reproducing Kernel Hilbert Space) on X if the evaluation functional $f \rightarrow f(x)$ is continuous for each $x \in X$. Let $\|\cdot\|_{\mathcal{F}}$ be a norm in \mathcal{F} and $c_{\mathcal{F}}(x) = \sup_{\|f\|_{\mathcal{F}} \leq 1} |f(x)|$. The embedding constant of \mathcal{F} is defined: $c_{\mathcal{F}} = \sup_x c_{\mathcal{F}}(x)$. We consider RKHS \mathcal{F} with $c_{\mathcal{F}} < \infty$.

Let $X = [0, 1]^m$ for $m \geq 1$. An example of RKHS is the Sobolev space $\mathcal{F} = H^1([0, 1])$, which consists of absolutely continuous functions $f : [0, 1] \rightarrow \mathcal{R}$ with $\|f\|_{\mathcal{F}} \leq 1$, where $\|f\|_{\mathcal{F}} = \sqrt{\int_0^1 (f(t))^2 dt + \int_0^1 (f'(t))^2 dt}$. For this space, $c_{\mathcal{F}} = \sqrt{\coth 1}$ (cf. Vovk 2005a).

Let \mathcal{F} be an RKHS on X with the dot product $(f \cdot g)$ for $f, g \in \mathcal{F}$. By Riesz–Fisher theorem, for each $x \in X$ there exists $k_x \in \mathcal{F}$ such that $f(x) = (k_x \cdot f)$.

The reproducing kernel is defined $K(x, y) = (k_x \cdot k_y)$. The main properties of the kernel:

1) $K(x, y) = K(y, x)$ for all $x, y \in X$ (symmetry property); 2) $\sum_{i,j=1}^k \alpha_i \alpha_j K(x_i, x_j) \geq 0$ for all k , for all $x_i \in X$, and for all real numbers α_i , where $i = 1, \dots, k$ (positive semidefinite property).

Conversely, a kernel defines RKHS: any symmetric, positive semidefinite kernel function $K(x, y)$ defines some canonical RKHS \mathcal{F} and a mapping $\Phi : X \rightarrow \mathcal{F}$ such that $K(x, y) = (\Phi(x) \cdot \Phi(y))$. Also, $c_{\mathcal{F}}(x) = \|k_x\|_{\mathcal{F}} = \|\Phi(x)\|_{\mathcal{F}}$. The mapping $\Phi(x)$ is also called "feature map" (cf. Cristianini and Shawe-Taylor 2000, Chapter 3).

A function $f : X \rightarrow \mathcal{R}$ is induced by a kernel $K(x, y)$ if there exists an element $g \in \mathcal{F}$ such that $f(x) = (g \cdot \Phi(x))$. This definition is independent of a map Φ . For any continuous kernel $K(x, y)$, every induced function f is continuous (cf. Steinwart (2001)).¹ In what follows we consider continuous kernels. Therefore, all functions from canonical RKHS \mathcal{F} are continuous.

For Sobolev space $H^1([0, 1])$, the reproducing kernel is

$$K(t, t') = (\cosh \min(t, t') \cosh \min(1 - t, 1 - t')) / \sinh 1$$

(cf. Vovk 2005a).

Well known examples of kernels on $X = [0, 1]^m$: Gaussian kernel $K(\bar{x}, \bar{y}) = \exp\{-\frac{\|\bar{x}-\bar{y}\|^2}{\sigma^2}\}$, where $\|\cdot\|$ is the Euclidian norm; $K(t, t') = \cos(\frac{\pi}{2}(t - t'))$, where $m = 1$ and $t, t' \in [0, 1]$.

Other examples and details of the kernel theory see in Scholkopf and Smola (2002).

Some special kernel corresponds to the method of randomization defined below. A random variable \tilde{y} is called randomization of a real number $y \in [0, 1]$ if $E(\tilde{y}) = y$, where E is the symbol of mathematical expectation with respect to the corresponding to \tilde{y} probability distribution.

We use a specific method of randomization of real numbers from unit interval proposed by Kakade and Foster (2004). Given positive integer number K divide the interval $[0, 1]$ on subintervals of length $\Delta = 1/K$ with rational endpoints $v_i = i\Delta$, where $i = 0, 1, \dots, K$. Let V denotes the set of these points. Any number $p \in [0, 1]$ can be represented as a linear combination of two neighboring endpoints of V defining subinterval containing p :

$$p = \sum_{v \in V} w_v(p)v = w_{v_{i-1}}(p)v_{i-1} + w_{v_i}(p)v_i, \quad (2)$$

where $p \in [v_{i-1}, v_i]$, $i = \lfloor p/\Delta \rfloor + 1$, $w_{v_{i-1}}(p) = 1 - (p - v_{i-1})/\Delta$, and $w_{v_i}(p) = (v_i - p)/\Delta$. Define $w_v(p) = 0$ for all other $v \in V$. Define a random variable

$$\tilde{p} = \begin{cases} v_{i-1} & \text{with probability } w_{v_{i-1}}(p) \\ v_i & \text{with probability } w_{v_i}(p) \end{cases}$$

Let $\bar{w}(p) = (w_v(p) : v \in V)$ be a vector of probabilities of rounding.

For any k -dimensional vector $\bar{x} = (x_1, \dots, x_k) \in [0, 1]^k$, we round each coordinate x_s , $s = 1, \dots, k$ to v_{j_s-1} with probability $w_{v_{j_s-1}}(x_s)$ and to v_{j_s} with probability $w_{v_{j_s}}(x_s)$, where $x_s \in [v_{j_s-1}, v_{j_s}]$. Let \tilde{x} be the corresponding random vector.

Let $v = (v^1, \dots, v^k) \in V^k$ and $W_v(\bar{x}) = \prod_{s=1}^k w_{v^s}(x_s)$. For any \bar{x} , let $\bar{W}(\bar{x}) = (W_v(\bar{x}) : v \in V^k)$ be a vector of probability distribution in V^k : $\sum_{v \in V^k} W_v(\bar{x}) = 1$. For $\bar{x}, \bar{y} \in [0, 1]^k$,

the dot product $K(\bar{x}, \bar{x}') = (\bar{W}(\bar{x}) \cdot \bar{W}(\bar{x}'))$ is the symmetric positive semidefinite kernel function.

3. Computing the well-calibrated forecasts

A universal trading strategy, which will be defined in Section 4, is based on the well-calibrated forecasts of stock prices. In this section we present a randomized algorithm for computing well-calibrated forecasts using a side information.

1. It is Lipschitz continuous (with respect to some semimetrics induced by the feature map (Steinwart 2001, Lemma 3)).

<p>Basic prediction protocol. FOR $i = 1, 2 \dots$ <i>Realty</i> announces a signal \mathbf{x}_i. <i>Predictor</i> announces a forecast p_i. <i>Realty</i> announces an outcome $y_i \in [0, 1]$. ENDFOR</p>
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Figure 1: Basic prediction protocol

A standard way to present any forecasting process is the perfect-information protocol (game). The most basic online perfect-information prediction protocol has two players *Realty* and *Predictor* (see Fig 1). In the perfect-information protocol, every player can see other players moves so far.

At the beginning of each step i , *Predictor* is given some data \mathbf{x}_i relevant to predicting the following outcome y_i . We call \mathbf{x}_i a signal or a side information. Signals are taken from the *object* space. Past outcomes and predictions are also known to *Predictor* in the perfect-information protocol.

The outcomes y_i are taken from an outcome space and predictions p_i are taken from a prediction space. In this paper an outcome is a real number from the unit interval $[0, 1]$ and a forecast is a single number from this interval (which can be output of a random variable), whereas Kakade and Foster considered a finite outcome space and a probability distribution on this space as a forecast. We could interpret the forecast p_i as the mean value of a future outcome y_i under some unknown to us probability distribution in $[0, 1]$.

In what follows we compare two types of forecasting algorithms: randomized algorithms which we will construct and stationary forecasting strategies which are continuous functions D from some RKHS using a side information as input. We consider two predictors D and C playing according to the basic prediction protocol (see Fig 1).

At the beginning of each step i *Predictor* D and *Predictor* C are given a signal \mathbf{x}_i . *Predictor* D uses a stationary prediction strategy $D(\mathbf{x}_i)$, where D is a function whose input is the signal \mathbf{x}_i . We suppose that \mathbf{x}_i is a real number from the unit interval. The number \mathbf{x}_i can encode any information. For example, it can be even the future outcome y_i .

Predictor C uses a randomized strategy which we will define below. We collect all information used for the internal randomization in a vector \bar{x}_i . This vector can contain any information known before the move of *Predictor* C at step i : the signal \mathbf{x}_i , past outcomes and so on.

For example, in Section 4, the information is one-dimensional vector $\bar{x}_i = y_{i-1}$ – the past outcome, in Section 6, $\bar{x}_i = (y_{i-1}, \mathbf{x}_i)$ is the pair of the past outcome and the signal.

In general, we suppose that \bar{x}_i is a vector of dimension $k \geq 1$: $\bar{x}_i \in [0, 1]^k$. We call it an *information* vector and assume that some method for computing information vectors given past outcomes and signals is fixed.

We use the tests of calibration to measure the discrepancy between predictions and outcomes. These tests are based on checking rules. We consider checking rules of more general type than that used in the literature on asymptotic calibration.

For any measurable subset $\mathcal{S} \subseteq [0, 1]^{k+1}$ define the checking rule

$$I_{\mathcal{S}}(p, \bar{x}) = \begin{cases} 1 & \text{if } (p, \bar{x}) \in \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases}$$

where \bar{x} is an k -dimensional vector.

In Section 3 we get $k = 1$ and $\mathcal{S} = \{(p, y) : p > y\}$ or $\mathcal{S} = \{(p, y) : p \leq y\}$, where $p, y \in [0, 1]$. In Section 6, $k = 2$ and a set \mathcal{S} is defined in a more complex way.

In the online prediction protocol defined on Fig 1, given $\Delta > 0$, a sequence of forecasts p_1, p_2, \dots is called Δ -calibrated for sequences of outcomes y_1, y_2, \dots and information vectors $\bar{x}_1, \bar{x}_2, \dots$ if the following asymptotic inequality

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n I_{\mathcal{S}}(p_i, \bar{x}_i)(y_i - p_i) \right| \leq \Delta$$

holds for all measurable subsets $\mathcal{S} \subseteq [0, 1]^{k+1}$. The sequence of forecasts is called *well-calibrated* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I_{\mathcal{S}}(p_i, \bar{x}_i)(y_i - p_i) = 0 \quad (3)$$

for all measurable subsets $\mathcal{S} \subseteq [0, 1]^{k+1}$.

If *Realty* acts adversatively, then, as shown by Oakes (1985) and Dawid (1985), any deterministic forecasting algorithm will not always be calibrated.

Following the method of Foster and Vohra (1998), at any step i , we will define a deterministic forecast p_i and randomize it to a random variable \tilde{p}_i using the sequential method of randomization defined in Section 2. We also randomize the information vector \bar{x}_i to a random vector \tilde{x}_i .

Let Pr be an overall probability distribution generated by this sequential method of randomization. We will show that for any measurable subset $S \subseteq [0, 1]^{k+1}$ with Pr -probability one the equality (3) is valid, where p_i and \bar{x}_i are replaced on their randomized variants \tilde{p}_i and \tilde{x}_i .

The following theorem on calibration with a side information is the main tool for an analysis presented in Sections 4 and 6.

In the perfect-information prediction protocol defined on Fig 1, let y_1, y_2, \dots be a sequence of outcomes and $\mathbf{x}_1, \mathbf{x}_2, \dots$ be the corresponding sequences of signals given online. We assume that a sequence of the information vectors $\bar{x}_1, \bar{x}_2, \dots \in \mathcal{R}^k$ also be defined online.

Let also, \mathcal{F} be an RKHS on $[0, 1]$ with a kernel $R(\mathbf{x}, \mathbf{x}')$ and a finite embedding constant $c_{\mathcal{F}}$.

Theorem 1 *For any $\epsilon > 0$, an algorithm for computing forecasts p_1, p_2, \dots and a sequential method of randomization can be constructed such that two conditions hold:*

- (i) *for any $\delta > 0$, with probability at least $1 - \delta$, for any measurable subset $S \subseteq [0, 1]^{k+1}$*

$$\left| \sum_{i=1}^n I_{\mathcal{S}}(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) \right| \leq 4e \left(\frac{k+1}{2} \right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{1 - \frac{1}{k+3} + \epsilon} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}} \quad (4)$$

for all n , where I_S is the characteristic function of S , $\tilde{p}_1, \tilde{p}_2, \dots$ are the corresponding randomizations of p_1, p_2, \dots and $\tilde{x}_1, \tilde{x}_2, \dots$ are the corresponding randomizations of k -dimensional information vectors $\bar{x}_1, \bar{x}_2, \dots$;

- (ii) for any $D \in \mathcal{F}$,

$$\left| \sum_{i=1}^n D(\mathbf{x}_i)(y_i - p_i) \right| \leq \|D\|_{\mathcal{F}} \sqrt{(c_{\mathcal{F}}^2 + 1)n} \quad (5)$$

for all n , where $\mathbf{x}_1, \mathbf{x}_2, \dots$ are signals.

Proof. At first, in Proposition 2 (below), given $\Delta > 0$, we modify a randomized rounding algorithm of Kakade and Foster (2004) to construct some Δ -calibrated forecasting algorithm, and combine it with Vovk (2005a) defensive forecasting algorithm. After that, we revise it by applying a variant of doubling trick argument such that (4) will hold.

Proposition 2 *Under the assumptions of Theorem 1, an algorithm for computing forecasts and a method of randomization can be constructed such that the inequality (5) holds for all D from RKHS \mathcal{F} and for all n . Also, for any $\delta > 0$ with probability at least $1 - \delta$,*

$$\left| \sum_{i=1}^n I_S(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) \right| \leq \Delta n + \sqrt{\frac{n(c_{\mathcal{F}}^2 + 1)}{\Delta^k}} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}$$

holds for all n , where I_S is the characteristic function of any measurable subset of $S \subseteq [0, 1]^{k+2}$.

Proof. We define a deterministic forecast and after that we randomize it.

The partition $V = \{v_0, \dots, v_K\}$ and probabilities of rounding were defined above by (2). In what follows we round some deterministic forecast p_n to v_{i-1} with probability $w_{v_{i-1}}(p_n)$ and to v_i with probability $w_{v_i}(p_n)$. We also round each coordinate $x_{n,s}$, $s = 1, \dots, k$, of the information vector \bar{x}_n to v_{j_s-1} with probability $w_{v_{j_s-1}}(x_{n,s})$ and to v_{j_s} with probability $w_{v_{j_s}}(x_{n,s})$, where $x_{n,s} \in [v_{j_s-1}, v_{j_s}]$.

Let $W_v(p_n, \bar{x}_n) = w_{v^1}(p_n)w_{v^2}(\bar{x}_n)$, where $v = (v^1, v^2)$ and $v^1 \in V$, $v^2 = (v_1^2, \dots, v_k^2) \in V^k$, $w_{v^2}(\bar{x}_n) = \prod_{s=1}^k w_{v_s^2}(x_{n,s})$, and $\bar{W}(p_n, \bar{x}_n) = (W_v(p_n, \bar{x}_n) : v \in V^{k+1})$ be a vector of probability distribution in V^{k+1} . Define the corresponding kernel $K(p, \bar{x}, p', \bar{x}') = (\bar{W}(p, \bar{x}) \cdot \bar{W}(p', \bar{x}'))$.

Let the deterministic forecasts p_1, \dots, p_{n-1} be already defined (put $p_1 = 1/2$). We want to define a deterministic forecast p_n .

The kernel $R(\mathbf{x}, \mathbf{x}')$ can be represented as a dot product in some feature space: $R(\mathbf{x}, \mathbf{x}') = (\Phi(\mathbf{x}) \cdot \Phi(\mathbf{x}'))$. Consider

$$U_n(p) = \sum_{i=1}^{n-1} (K(p, \bar{x}_n, p_i, \bar{x}_i) + R(\mathbf{x}_n, \mathbf{x}_i))(y_i - p_i). \quad (6)$$

The following lemma presents a general method for computing deterministic forecasts.

Define $\mathcal{M}_0 = 1$ and

$$\mathcal{M}_n = \mathcal{M}_{n-1} + U_n(p_n)(y_n - p_n)$$

for all n .

Lemma 3 (Vovk et al. 2005) *A sequence of forecasts p_1, p_2, \dots can be computed such that $\mathcal{M}_n \leq \mathcal{M}_{n-1}$ for all n .*

Proof. By definition the function $U_n(p)$ is continuous in p . The needed forecast is computed as follows. If $U_n(p) > 0$ for all $p \in [0, 1]$ then define $p_n = 1$; if $U_n(p) < 0$ for all $p \in [0, 1]$ then define $p_n = 0$. Otherwise, define p_n to be a root of the equation $U_n(p) = 0$ (some root exists by the intermediate value theorem). Evidently, $\mathcal{M}_n \leq \mathcal{M}_{n-1}$ for all n . Lemma is proved.

Let forecasts p_1, p_2, \dots be computed by the method of Lemma 3. Then for any N ,

$$\begin{aligned}
0 &\geq \mathcal{M}_N - \mathcal{M}_0 = \sum_{n=1}^N U_n(p_n)(y_n - p_n) = \\
&= \sum_{n=1}^N \sum_{i=1}^{n-1} (K(p_n, \bar{x}_n, p_i, \bar{x}_i) + R(\mathbf{x}_n, \mathbf{x}_i))(y_i - p_i)(y_n - p_n) = \\
&= \frac{1}{2} \sum_{n=1}^N \sum_{i=1}^N K(p_n, \bar{x}_n, p_i, \bar{x}_i)(y_i - p_i)(y_n - p_n) - \\
&\quad - \frac{1}{2} \sum_{n=1}^N (K(p_n, \bar{x}_n, p_n, \bar{x}_n)(y_n - p_n))^2 + \\
&\quad + \frac{1}{2} \sum_{n=1}^N \sum_{i=1}^N R(\mathbf{x}_n, \mathbf{x}_i)(y_i - p_i)(y_n - p_n) - \\
&\quad - \frac{1}{2} \sum_{n=1}^N (R(\mathbf{x}_n, \mathbf{x}_n)(y_n - p_n))^2 = \tag{7}
\end{aligned}$$

$$= \frac{1}{2} \left\| \sum_{n=1}^N \bar{W}(p_n, \bar{x}_n)(y_n - p_n) \right\|^2 - \frac{1}{2} \sum_{n=1}^N \|\bar{W}(p_n, \bar{x}_n)\|^2 (y_n - p_n)^2 + \tag{8}$$

$$+ \frac{1}{2} \left\| \sum_{n=1}^N \Phi(\mathbf{x}_n)(y_n - p_n) \right\|_{\mathcal{F}}^2 - \frac{1}{2} \sum_{n=1}^N \|\Phi(\mathbf{x}_n)\|_{\mathcal{F}}^2 (y_n - p_n)^2. \tag{9}$$

In (8), $\|\cdot\|$ is Euclidian norm, and in (9), $\|\cdot\|_{\mathcal{F}}$ is the norm in RKHS \mathcal{F} .

Since $(y_n - p_n)^2 \leq 1$ for all n and

$$\|(\bar{W}(p_n, \bar{x}_n))\|^2 = \sum_{v \in V^{k+1}} (W_v(p_n, \bar{x}_n))^2 \leq \sum_{v \in V^{k+1}} W_v(p_n, \bar{x}_n) = 1,$$

the subtracted sum of (8) is upper bounded by N .

Since $\|\Phi(\mathbf{x}_n)\|_{\mathcal{F}} = c_{\mathcal{F}}(\bar{x}_n)$ and $c_{\mathcal{F}}(\mathbf{x}) \leq c_{\mathcal{F}}$ for all \mathbf{x} , the subtracted sum of (9) is upper bounded by $c_{\mathcal{F}}^2 N$. As a result we obtain

$$\left\| \sum_{n=1}^N \bar{W}(p_n, \bar{x}_n)(y_n - p_n) \right\| \leq \sqrt{(c_{\mathcal{F}}^2 + 1)N} \tag{10}$$

$$\left\| \sum_{n=1}^N \Phi(\mathbf{x}_n)(y_n - p_n) \right\|_{\mathcal{F}} \leq \sqrt{(c_{\mathcal{F}}^2 + 1)N} \tag{11}$$

for all N . Let us denote $\bar{\mu}_n = \sum_{i=1}^n \bar{W}(p_i, \bar{x}_i)(y_i - p_i)$. By (10), $\|\bar{\mu}_n\| \leq \sqrt{(c_{\mathcal{F}}^2 + 1)n}$ for all n .

Let $\bar{\mu}_n = (\mu_n(v) : v \in V^{k+1})$. By definition for any v

$$\mu_n(v) = \sum_{i=1}^n W_v(p_i, \bar{x}_i)(y_i - p_i). \quad (12)$$

Insert the term $I(v)$ in the sum (12), where I is the characteristic function of an arbitrary set $\mathcal{S} \subseteq [0, 1]^{k+1}$, sum by $v \in V^{k+1}$, and exchange the order of summation. Using Cauchy–Schwarz inequality for vectors $\bar{I} = (I(v) : v \in V^{k+1})$, $\bar{\mu}_n = (\mu_n(v) : v \in V^{k+1})$ and Euclidian norm, we obtain

$$\begin{aligned} & \left| \sum_{i=1}^n \sum_{v \in V^{k+1}} W_v(p_i, \bar{x}_i) I(v)(y_i - p_i) \right| = \\ & = \left| \sum_{v \in V^{k+1}} I(v) \sum_{i=1}^n W_v(p_i, \bar{x}_i)(y_i - p_i) \right| = \\ & = (\bar{I} \cdot \bar{\mu}_n) \leq \|\bar{I}\| \cdot \|\bar{\mu}_n\| \leq \sqrt{|V^{k+1}|(c_{\mathcal{F}}^2 + 1)n} \end{aligned} \quad (13)$$

for all n , where $|V^{k+1}| = 1/\Delta^{k+1}$ – is the cardinality of the partition.

Let \tilde{p}_i be a random variable taking values $v \in V$ with probabilities $w_v(p_i)$ (only two of them are nonzero). Recall that \tilde{x}_i is a random variable taking values $v \in V^k$ with probabilities $w_v(\bar{x}_i)$.

Let $\mathcal{S} \subseteq [0, 1]^{k+1}$ and I be its indicator function. For any i , the mathematical expectation of a random variable $I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)$ is equal to

$$E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) = \sum_{v \in V^{k+1}} W_v(p_i, \bar{x}_i) I(v)(y_i - v^1), \quad (14)$$

where $v = (v^1, v^2)$. By Azuma–Hoeffding inequality (see (26) below), for any $\delta > 0$, with Pr -probability $1 - \delta$,

$$\left| \sum_{i=1}^n I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) - \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}. \quad (15)$$

By definition of the deterministic forecast

$$\left| \sum_{v \in V^{k+1}} W_v(p_i, \bar{x}_i) I(v)(y_i - p_i) - \sum_{v \in V^{k+1}} W_v(p_i, \bar{x}_i) I(v)(y_i - v^1) \right| < \Delta$$

for all i , where $v = (v^1, v^2)$. Summing (14) by $i = 1, \dots, n$ and using the inequality (13), we obtain

$$\left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| =$$

$$\begin{aligned}
&= \left| \sum_{i=1}^n \sum_{v \in V^{k+1}} W_v(p_i, \bar{x}_i) I(v)(y_i - v^1) \right| < \\
&< \Delta n + \sqrt{(c_{\mathcal{F}}^2 + 1)n/\Delta^{k+1}}
\end{aligned} \tag{16}$$

for all n .

By (15) and (16), with Pr -probability $1 - \delta$,

$$\left| \sum_{i=1}^n I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) \right| \leq \Delta n + \sqrt{(c_{\mathcal{F}}^2 + 1)n/\Delta^{k+1}} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}. \tag{17}$$

By Cauchy–Schwarz inequality

$$\begin{aligned}
\left| \sum_{n=1}^N D(\bar{x}_n)(y_n - p_n) \right| &= \left| \sum_{n=1}^N (y_n - p_n)(D \cdot \Phi(\bar{x}_n)) \right| = \\
\left| \left(\sum_{n=1}^N (y_n - p_n) \Phi(\bar{x}_n) \cdot D \right) \right| &\leq \left\| \sum_{n=1}^N (y_n - p_n) \Phi(\bar{x}_n) \right\|_{\mathcal{F}} \cdot \|D\|_{\mathcal{F}} \leq \\
&\leq \|D\|_{\mathcal{F}} \sqrt{(c_{\mathcal{F}}^2 + 1)N}.
\end{aligned}$$

Proposition is proved.

Now we turn to the proof of Theorem 1.

The expression $\Delta n + \sqrt{(c_{\mathcal{F}}^2 + 1)n/\Delta^{k+1}}$ from (16) and (17) takes its minimal value for $\Delta = \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{-\frac{1}{k+3}}$. In this case, the right-hand side of the inequality (16) is equal to

$$\Delta n + \sqrt{n(c_{\mathcal{F}}^2 + 1)/\Delta^{k+1}} \leq 2\Delta n = 2 \left(\frac{k+1}{2} \right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{1-\frac{1}{k+3}}. \tag{18}$$

In what follows we use the upper bound $2\Delta n$ in (16).

To prove the bound (4) choose a monotonic sequence of rational numbers $\Delta_1 > \Delta_2 > \dots$ such that $\Delta_s \rightarrow 0$ as $s \rightarrow \infty$. We also define an increasing sequence of positive integer numbers $n_1 < n_2 < \dots$. For any s , we use for randomization on steps $n_s \leq n < n_{s+1}$ the partition of $[0, 1]$ on subintervals of length Δ_s .

We start our sequences from $n_1 = 1$ and $\Delta_1 = 1$. Also, define the numbers n_2, n_3, \dots such that the inequality

$$\left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq 4(s+1)\Delta_s n \tag{19}$$

holds for all $n_s \leq n \leq n_{s+1}$ and for all $s \geq 1$.

We define this sequence by mathematical induction on s . Suppose that n_s ($s \geq 1$) is defined such that the inequality

$$\left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq 4s\Delta_{s-1} n \tag{20}$$

holds for all $n_{s-1} \leq n \leq n_s$, and the inequality

$$\left| \sum_{i=1}^{n_s} E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq 4s\Delta_s n_s \quad (21)$$

also holds.

Let us define n_{s+1} . Consider all forecasts \tilde{p}_i defined by the algorithm given above for the discretization $\Delta = \Delta_{s+1}$. We do not use first n_s of these forecasts (more correctly we will use them only in bounds (22) and (23); denote these forecasts $\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_{n_s}$). We add the forecasts \tilde{p}_i for $i > n_s$ to the forecasts defined before this step of induction (for n_s). Let n_{s+1} be such that the inequality

$$\begin{aligned} & \left| \sum_{i=1}^{n_{s+1}} E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq \left| \sum_{i=1}^{n_s} E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| + \\ & + \left| \sum_{i=n_s+1}^{n_{s+1}} E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) + \sum_{i=1}^{n_s} E(I(\hat{\mathbf{p}}_i, \tilde{x}_i)(y_i - \hat{\mathbf{p}}_i)) \right| + \\ & + \left| \sum_{i=1}^{n_s} E(I(\hat{\mathbf{p}}_i, \tilde{x}_i)(y_i - \hat{\mathbf{p}}_i)) \right| \leq 4(s+1)\Delta_{s+1}n_{s+1} \end{aligned} \quad (22)$$

holds. Here the first sum of the right-hand side of the inequality (22) is bounded by $4s\Delta_s n_s$ – by the induction hypothesis (21). The second and third sums are bounded by $2\Delta_{s+1}n_{s+1}$ and by $2\Delta_{s+1}n_s$, respectively, where $\Delta = \Delta_{s+1}$ is defined such that (18) holds. This follows from (16) and by choice of n_s .

The induction hypothesis (21) is valid for

$$n_{s+1} \geq \frac{2s\Delta_s + \Delta_{s+1}}{\Delta_{s+1}(2s+1)} n_s.$$

Similarly,

$$\begin{aligned} & \left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq \left| \sum_{i=1}^{n_s} E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| + \\ & + \left| \sum_{i=n_s+1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) + \sum_{i=1}^{n_s} E(I(\hat{\mathbf{p}}_i, \tilde{x}_i)(y_i - \hat{\mathbf{p}}_i)) \right| + \\ & + \left| \sum_{i=1}^{n_s} E(I(\hat{\mathbf{p}}_i, \tilde{x}_i)(y_i - \hat{\mathbf{p}}_i)) \right| \leq 4(s+1)\Delta_s n \end{aligned} \quad (23)$$

for $n_s < n \leq n_{s+1}$. Here the first sum of the right-hand inequality (22) is also bounded by $4s\Delta_s n_s \leq 4s\Delta_s n$ – by the induction hypothesis (21). The second and the third sums are bounded by $2\Delta_{s+1}n \leq 2\Delta_s n$ and by $2\Delta_{s+1}n_s \leq 2\Delta_s n$, respectively. This follows from (16) and from choice of Δ_s . The induction hypothesis (20) is valid.

By (19) for any s

$$\left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq 4(s+1)\Delta_s n \quad (24)$$

for all $n \geq n_s$ if Δ_s satisfies the condition $\Delta_{s+1} \leq \Delta_s(1 - \frac{1}{s+2})$ for all s .

We show now that sequences n_s and Δ_s satisfying all the conditions above exist.

Let $\epsilon > 0$ and $M = \lceil 1/\epsilon \rceil$, where $\lceil r \rceil$ is the least integer number such that $m \geq r$. Define $n_s = (s + M)^M$ and $\Delta_s = \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n_s^{-\frac{1}{k+3}}$. Easy to verify that all requirements for n_s and Δ_s given above are satisfied, where ϵ is sufficiently small.

We have in (24) for all $n_s \leq n < n_{s+1}$

$$\begin{aligned} 4(s+1)\Delta_s n &\leq 4(s+M)\Delta_s n_{s+1} = \\ &= 4 \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} (s+M)(s+M+1)^M (s+M)^{-\frac{M}{k+3}} \leq \\ &\leq 4e \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n_s^{1-\frac{1}{k+3}+2/M} \leq \\ &\leq 4e \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{1-\frac{1}{k+3}+\epsilon}, \end{aligned}$$

where e is the base of the natural logarithm. Therefore, we obtain

$$\left| \sum_{i=1}^n E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i)) \right| \leq 4e \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{1-\frac{1}{k+3}+\epsilon} \quad (25)$$

for all n . Azuma–Hoeffding inequality says that for any $\gamma > 0$

$$Pr \left\{ \left| \frac{1}{n} \sum_{i=1}^n V_i \right| > \gamma \right\} \leq 2e^{-2n\gamma^2} \quad (26)$$

for all n , where V_i are martingale–differences.

We get $V_i = I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) - E(I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i))$ and $\gamma = \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}$, where $\delta > 0$.

Denote $\nu(n) = 4e \left(\frac{k+1}{2}\right)^{\frac{2}{k+3}} (c_{\mathcal{F}}^2 + 1)^{\frac{1}{k+3}} n^{1-\frac{1}{k+3}+\epsilon}$.

Combining (25) with (26), we obtain that for any $\delta > 0$, with probability $1 - \delta$,

$$\left| \sum_{i=1}^n I(\tilde{p}_i, \tilde{x}_i)(y_i - \tilde{p}_i) \right| \leq \nu(n) + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}$$

for all n . Theorem 1 is proved.

4. Competing with stationary trading strategies from RKHS

A trading game has two players: *Trader* and *Stock Market*. They correspond to *Predictor* and *Reality* in the simple prediction game defined in Section 3.

We suppose that the prices S_1, S_2, \dots of a stock are bounded and rescaled such that $0 \leq S_i \leq 1$ for all t . We get also $S_0 = 0$. These prices are analogs of outcomes of the prediction game.

We present the process of online trading in Stock Market in the form of a trading game regulated by the perfect-information protocol presented on Fig 2.

Basic trading protocol.

FOR $i = 1, 2 \dots$

Stock Market announces a signal $\mathbf{x}_i \in X$.

Trader bets by buying or selling a number C_i of shares of the stock by S_{i-1} each.

Stock Market reveals a price S_i of the stock.

Trader receives his total gain (or suffers loss) at the end of step i :

$\mathcal{K}_i = \mathcal{K}_{i-1} + C_i(S_i - S_{i-1})$. We get $\mathcal{K}_0 = 0$.

ENDFOR

Figure 2: Basic trading protocol

At the beginning of each step i *Trader* is given an object $\mathbf{x}_i \in X$ which was called a side information at step i . Without loss of generality suppose that $X = [0, 1]$.

We call the sequence C_i a *trading strategy*. In case $C_i > 0$ *Trader* playing for a rise, in case $C_i < 0$ *Trader* playing for a fall, *Trader* passes the step if $C_i = 0$. We suppose that *Trader* can borrow money for buying C_i shares and can incur debt.

A *stationary trading strategy* is a function D from X to \mathcal{R} . We suppose that some RKHS \mathcal{F} on X with a kernel $R(\mathbf{x}, \mathbf{x}')$ and with a finite embedding constant $c_{\mathcal{F}}$ is given.

Any stationary trading strategy D uses at step i a side information – a real number $\mathbf{x}_i \in X$.

Our universal trading strategy will be randomized. By a randomized trading strategy we mean a sequence \tilde{M}_i , $i = 1, 2, \dots$, of the random variables.

The universal trading strategy which we define below uses the past price S_{i-1} of the stock as one-dimensional information vector in sense of Theorem 1, where $S_0 = 0$. This information is used for the internal randomization.

We define a universal trading strategy as a random variable \tilde{M}_i and show that this trading strategy performs almost surely at least as good as any stationary trading strategy $D \in \mathcal{F}$ using arbitrary side information \mathbf{x}_i .

To be more concise, define on Fig 3 the perfect-information protocol of the game with two traders: *Trader M* uses the randomized strategy \tilde{M}_i , *Trader D* uses an arbitrary stationary trading strategy $D \in \mathcal{F}$.

This protocol is more general than two basic trading protocols (Fig 2) together, since *Stock Market* can use information on the decisions of both traders M and D before revealing a future price S_i .

At first, for simplicity, we consider a case of dealing for a rise, since the proof of optimality (Theorem 4) is much more clear in this case than that in general case (Theorem 5). Also, a series of numerical experiments presented in Section 7, are performed for the case where both traders dealing for a rise. The case of dealing for a fall is considered similarly.

At each step i we will compute a forecast p_i of a future price and randomize it to \tilde{p}_i . We also randomize the past price S_{i-1} of the stock to \tilde{S}_{i-1} . Details of this computation and randomization are given in Section 3. Our universal strategy is a randomized decision rule – it takes only two values:

$$\tilde{M}_i^1 = \begin{cases} 1 & \text{if } \tilde{p}_i > \tilde{S}_{i-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Trading protocol with two traders.

FOR $i = 1, 2 \dots$

Stock Market announces a signal \mathbf{x}_i .

Trader M bets by buying or selling the random number \tilde{M}_i of shares of the stock by S_{i-1} each.

Trader D bets by buying or selling a number $D(\mathbf{x}_i)$ of shares of the stock by S_{i-1} each.

Stock Market reveals a price S_i of the stock.

Trader M receives his total gain (or suffers loss) at the end of step i :

$\mathcal{K}_i^M = \mathcal{K}_{i-1}^M + \tilde{M}_i(S_i - S_{i-1})$. We get $\mathcal{K}_0^M = 0$.

Trader D receives his total gain (or suffers loss) at the end of step i :

$\mathcal{K}_i^D = \mathcal{K}_{i-1}^D + D(\mathbf{x}_i)(S_i - S_{i-1})$. We get $\mathcal{K}_0^D = 0$.

ENDFOR

Figure 3: Trading protocol with two traders

Assume that prices $S_1, S_2, \dots \in [0, 1]$ and signals $\mathbf{x}_1, \mathbf{x}_2, \dots \in [0, 1]$ be given online according to the protocol presented on Fig 3. Denote $\Delta S_i = S_i - S_{i-1}$. We use the norm

$$\|D\|_\infty = \sup_{\mathbf{x} \in [0, 1]} |D(\mathbf{x})|,$$

where D is a nonnegative continuous function. If a function D is not identically zero then $\|D\|_\infty > 0$. We call such a function nontrivial.

Informally, our main result says that if the forecasts \tilde{p}_i are well-calibrated on the sequence of prices S_i , $i = 1, 2, \dots$, then *Trader M* performs at least as good as any *Trader D* playing for a rise.

Theorem 4 *An algorithm for computing forecasts p_i and a sequential method of randomization can be constructed such that for any nontrivial nonnegative stationary trading strategy $D \in \mathcal{F}$*

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i^1 \Delta S_i - \frac{1}{n} \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i \right) \geq 0 \quad (27)$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

Moreover, for any $\epsilon > 0$ this trading strategy \tilde{M}^1 can be tuned such that for any $\delta > 0$, with probability at least $1 - \delta$, for all nontrivial nonnegative $D \in \mathcal{F}$

$$\begin{aligned} \sum_{i=1}^n \tilde{M}_i^1 \Delta S_i &\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i - \\ &- \frac{4}{3} (7e - 1) (c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}} n^{\frac{3}{4} + \epsilon} - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}} \sqrt{(c_{\mathcal{F}}^2 + 1)n} - \\ &- \sqrt{\frac{n}{2} \ln \frac{2}{\delta}} \end{aligned} \quad (28)$$

for all n , where e is the base of the natural logarithm.

Proof. We use the randomized trading strategy \tilde{M}^1 based on the well-calibrated forecasts defined in Section 3, where $y_i = S_i$ and $\bar{x}_i = S_{i-1}$.

Recall that $\epsilon > 0$ and $M = \lceil 1/\epsilon \rceil$. At any step i we compute the deterministic forecast p_i defined in Section 3 and its randomization to \tilde{p}_i using parameters $\Delta = \Delta_s = (c_{\mathcal{F}} + 1)^{\frac{1}{4}}(s + M)^{-\frac{M}{4}}$ and $n_s = (s + M)^M$, where $n_s \leq i < n_{s+1}$. Let also, \tilde{S}_{i-1} be a randomization of the past price S_{i-1} . The following upper bound directly follows from the method of discretization:

$$\begin{aligned} \left| \sum_{i=1}^n I(\tilde{p}_i > \tilde{S}_{i-1})(\tilde{S}_{i-1} - S_{i-1}) \right| &\leq \sum_{t=0}^s (n_{t+1} - n_t) \Delta_t \leq \\ &\leq \frac{4}{3}(e-1)(c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}} n_s^{\frac{3}{4} + \epsilon} \leq \frac{4}{3}(e-1)(c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}} n^{\frac{3}{4} + \epsilon}. \end{aligned} \quad (29)$$

Let $D(\mathbf{x})$ be an arbitrary nontrivial nonnegative trading strategy from RKHS \mathcal{F} . Clearly, the bound (29) holds if we replace $I(\tilde{p}_i > \tilde{S}_{i-1})$ on $\|D\|^{-1}D(\mathbf{x}_i)$.

Let \tilde{M}^1 be the randomized trading strategy defined above. We use abbreviations:

$$\nu_1(n) = (c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}}(e-1)\frac{4}{3}n^{\frac{3}{4} + \epsilon}, \quad (30)$$

$$\nu_2(n) = 4en^{\frac{3}{4} + \epsilon}(c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}, \quad (31)$$

$$\nu_3(n) = \sqrt{(c_{\mathcal{F}}^2 + 1)n} \quad (32)$$

All sums below are for $i = 1, \dots, n$. By definition $0 \leq D(\mathbf{x}_i) \leq \|D\|_{\infty}$ for all $\mathbf{x}_i \in [0, 1]$.

Let $\delta > 0$. Then, with probability $1 - \delta$, for any $D \in \mathcal{F}$, the following chain of equalities and inequalities is valid:

$$\begin{aligned} &\sum_{i=1}^n \tilde{M}_i^1(S_i - S_{i-1}) = \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - S_{i-1}) = \\ &= \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - \tilde{p}_i) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{S}_{i-1} - S_{i-1}) \geq \end{aligned} \quad (33)$$

$$\geq \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) \geq \quad (34)$$

$$\geq \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) = \quad (35)$$

$$\begin{aligned} &= \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(p_i - S_{i-1}) + \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{p}_i - p_i) - \\ &- \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{S}_{i-1} - S_{i-1}) - \nu_1(n) - \nu_2(n) \geq \end{aligned} \quad (36)$$

$$\geq \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(p_i - S_{i-1}) - 3\nu_1(n) - \nu_2(n) \geq \quad (37)$$

$$\begin{aligned} &\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - S_{i-1}) - \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - p_i) - \\ &\quad - 4\nu_1(n) - \nu_2(n) - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}} \nu_3(n) = \end{aligned} \quad (38)$$

$$\begin{aligned} &= \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - S_{i-1}) - \\ &\quad - 4\nu_1(n) - \nu_2(n) - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}} \nu_3(n). \end{aligned} \quad (39)$$

In transition from (33) to (34) the inequality (4) of Theorem 1 and the bound (29) were used, and so, the terms (30) and (31) were subtracted. The transition from (34) to (35) is valid since $0 \leq D(\mathbf{x}) \leq \|D\|_\infty$ for all \mathbf{x} . In transition from (36) to (37) the bound (29) was applied twice to intermediate terms, and so, the term (29) was subtracted twice. In transition from (37) to (38) the inequality (5) of Theorem 1 was used, and so, the term (32) was subtracted. In transition from (38) to (39) we have used the inequality (5) of Theorem 1. Therefore, we have (28).

The inequality (27) follows from (28). Theorem 4 is proved.

Now, we consider the general case of dealing for a rise and for a fall. The corresponding trading strategy is defined:

$$\tilde{M}_i = \begin{cases} 1 & \text{if } \tilde{p}_i > \tilde{S}_{i-1}, \\ -1 & \text{if } \tilde{p}_i \leq \tilde{S}_{i-1}. \end{cases}$$

Trader D is also dealing for a rise and for a fall.

Let $S_1, S_2, \dots \in [0, 1]$ and $\mathbf{x}_1, \mathbf{x}_2, \dots \in [0, 1]$ be given online according to the protocol presented on Fig 3.

Theorem 5 *An algorithm for computing forecasts p_i and a sequential method of randomization can be constructed such that for any nontrivial stationary trading strategy $D \in \mathcal{F}$,*

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{n} \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i \right) \geq 0 \quad (40)$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

Moreover, for any $\epsilon > 0$ this trading strategy M can be tuned such that for any $\delta > 0$, with probability at least $1 - \delta$, for all nontrivial $D \in \mathcal{F}$

$$\begin{aligned} \sum_{i=1}^n \tilde{M}_i \Delta S_i &\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i - \\ &- \frac{8}{3} (5e - 2) (c_{\mathcal{F}}^2 + 1)^{\frac{1}{4}} n^{\frac{3}{4} + \epsilon} - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}} \sqrt{(c_{\mathcal{F}}^2 + 1)n} - \\ &\quad - 2 \sqrt{\frac{n}{2} \ln \frac{2}{\delta}} \end{aligned} \quad (41)$$

for all n .

Proof. We use abbreviations (30)–(32) from the proof of Theorem 4.

Define

$$D^+(\mathbf{x}) = \begin{cases} D(\mathbf{x}) & \text{if } D(\mathbf{x}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$D^-(\mathbf{x}) = \begin{cases} D(\mathbf{x}) & \text{if } D(\mathbf{x}) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

By definition $D(\mathbf{x}) = D^+(\mathbf{x}) + D^-(\mathbf{x})$.

The proof of Theorem 5 is based on transformations similar to (33)–(39).

Let $\delta > 0$. Then with probability $1 - \delta$ for any $D \in \mathcal{F}$:

$$\begin{aligned} & \sum_{i=1}^n \tilde{M}_i(S_i - S_{i-1}) = \\ & = \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - S_{i-1}) - \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} (S_i - S_{i-1}) = \\ & = \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - \tilde{p}_i) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{S}_{i-1} - S_{i-1}) - \\ & - \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} (S_i - \tilde{p}_i) - \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) - \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} (\tilde{S}_{i-1} - S_{i-1}) \geq \\ & \geq \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) - \end{aligned} \quad (42)$$

$$- \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) \geq \quad (43)$$

$$\geq \|D\|_\infty^{-1} \sum_{\tilde{p}_i > \tilde{S}_{i-1}} D^+(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) + \quad (44)$$

$$+ \|D\|_\infty^{-1} \sum_{\tilde{p}_i \leq \tilde{S}_{i-1}} D^-(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) = \quad (45)$$

$$\begin{aligned} & \geq \|D\|_\infty^{-1} \sum_{i=1}^n D^+(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) + \\ & + \|D\|_\infty^{-1} \sum_{i=1}^n D^-(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) = \\ & = \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - 2\nu_1(n) - 2\nu_2(n) = \\ & = \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(p_i - S_{i-1}) + \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{p}_i - p_i) - \\ & - \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(\tilde{S}_{i-1} - S_{i-1}) - 2\nu_1(n) - 2\nu_2(n) \geq \end{aligned}$$

$$\begin{aligned}
&\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(p_i - S_{i-1}) - 4\nu_1(n) - 2\nu_2(n) \geq \\
&\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - S_{i-1}) - \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - p_i) - \\
&\quad - 4\nu_1(n) - 2\nu_2(n) - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}\nu_3}(n) = \\
&\quad = \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i)(S_i - S_{i-1}) - \\
&\quad - 4\nu_1(n) - 2\nu_2(n) - \|D\|_\infty^{-1} \|D\|_{\mathcal{F}\nu_3}(n).
\end{aligned}$$

The proof of these transitions is similar to the proof of transitions in (33)–(39) of Theorem 4. This completes the proof of Theorem 5.

Theorem 5 can be rewritten for the strategy $\tilde{M}_i^l = l\tilde{M}_i$ and for the class of stationary strategies $D \in \mathcal{F}$ with bounded norm $\|D\|_\infty \leq l$, where l is an arbitrary positive integer number.

We present the following statement for \tilde{M}_i^l .

Corollary 6 *An algorithm for computing forecasts p_i and a sequential method of randomization can be constructed such that given complexity bound $l > 0$ for any nontrivial stationary trading strategy $D \in \mathcal{F}$ such that $\|D\|_\infty \leq l$*

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i^l \Delta S_i - \frac{1}{n} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i \right) \geq 0$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

For any $\epsilon > 0$, this trading strategy \tilde{M}_i^l can be tuned such that for any $\delta > 0$, with probability at least $1 - \delta$, for all nonnegative $D \in \mathcal{F}$ such that $\|D\|_\infty \leq l$ and for all n

$$\begin{aligned}
&\sum_{i=1}^n \tilde{M}_i^l \Delta S_i \geq \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i - \\
&-\frac{8}{3}(5e-2)l(c_{\mathcal{F}}^2+1)^{\frac{1}{4}}n^{\frac{3}{4}+\epsilon} - \|D\|_{\mathcal{F}}\sqrt{(c_{\mathcal{F}}^2+1)n} - 2l\sqrt{\frac{n}{2}\ln\frac{2}{\delta}}.
\end{aligned}$$

5. Universal consistency

Using a universal kernel and the corresponding canonical universal RKHS, we can extend our asymptotic results for all continuous stationary trading strategies.

An RKHS \mathcal{F} on X is universal if X is a compact metric space and every continuous function f on X can be arbitrarily well approximated in the metric $\|\cdot\|_\infty$ by a function from \mathcal{F} : for any $\epsilon > 0$ there exists $D \in \mathcal{F}$ such that

$$\sup_{x \in X} |f(x) - D(x)| \leq \epsilon$$

(cf. Steinwart 2001, Definition 4).

We use $X = [0, 1]$. The Sobolev space $\mathcal{F} = H^1([0, 1])$ defined in Section 2 is the universal RKHS (cf. Steinwart 2001, Vovk 2005a).

We call a randomized trading strategy \tilde{M}_i universally consistent if for any continuous function f with probability one

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i(S_i - S_{i-1}) - \frac{1}{n} \|f\|_\infty^{-1} \sum_{i=1}^n f(\mathbf{x}_i)(S_i - S_{i-1}) \right) \geq 0. \quad (46)$$

This definition is similar to Vovk (2005a) definition of a universally consistent prediction strategy.

The existence of the universal RKHS on $[0, 1]$ implies the following

Theorem 7 *An algorithm for computing forecasts p_i and a sequential method of randomization can be constructed which performs at least as good as any nontrivial continuous trading strategy f :*

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{n} \|f\|_\infty^{-1} \sum_{i=1}^n f(\mathbf{x}_i) \Delta S_i \right) \geq 0 \quad (47)$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

This result directly follows from the possibility to approximate arbitrarily close any continuous function f on $[0, 1]$ by a function D from the universal RKHS \mathcal{F} : for any nontrivial continuous function f and for any $0 < \epsilon < 1$ take a nontrivial $D \in \mathcal{F}$ such that $\|f - D\|_\infty < \frac{1}{5}\epsilon\|f\|_\infty$. Then

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{n} \|f\|_\infty^{-1} \sum_{i=1}^n f(\mathbf{x}_i) \Delta S_i \right) + \epsilon \geq \\ & \geq \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{n} \|D\|_\infty^{-1} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i \right) \geq 0. \end{aligned} \quad (48)$$

Since (48) holds for each $\epsilon > 0$, (47) is valid.

The property of universal consistency is asymptotic and does not tell us anything about finite data sequences: we cannot obtain the convergence bounds like (28) and (41) which holds for stationary strategies from RKHS.

6. Competing with discontinuous trading strategies

The trading strategy \tilde{M}_i defined in Section 4 performs at least as good as any stationary trading strategy $D(\mathbf{x})$ (up to some regret) even if the future price S_i of the stock is known to D as a side information contained in \mathbf{x}_i . Theorems 4 and 5 are also valid in this case.

This impressive efficiency of the trading strategy \tilde{M}_i can be explained by the restrictive power of continuous functions. A lack of *Trader D* is that a set of his strategies is limited by \mathcal{F} . A continuous stationary trading strategy D cannot respond sufficiently quickly to

information about changes of the value of a future price S_i . the optimal trading strategy \tilde{M}_i , is a discontinuous function, though it is applied to the random variables.

A positive argument in favor of the requirement of continuity of D is that it is natural to compete only with computable trading strategies, and continuity is often regarded as a necessary condition for computability (Brouwer's "continuity principle").

If D is allowed to be discontinuous, we cannot prove (27) and (40) in general case.

Let \tilde{M}_i be an arbitrary randomizing trading strategy. For simplicity, we assume that $\tilde{M}_1, \tilde{M}_2, \dots$ is a sequence of i.i.d. random variables.²

A stationary trading strategy $D(\mathbf{x})$ is called *decision rule* if its range is finite. Decision rule is binary if it takes only two values.

Theorem 8 *Let \tilde{M}_i be an arbitrary i.i.d sequence of random variables (randomized trading strategy) such that $|\tilde{M}_i| \leq 1$ for all i .*

Consider the protocol of trading game with two players and with signals $\mathbf{x}_i = P\{\tilde{M}_i > 0\}$ for all i presented on Fig 3.

Then a binary decision rule $D(\mathbf{x})$ and a sequence S_1, S_2, \dots of prices can be defined such that with probability one

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{2} \frac{1}{n} \sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i \right) \leq 0, \quad (49)$$

where $\Delta S_i = S_i - S_{i-1}$. Inequality (49) means that trading strategy D outperforms \tilde{M}_i twice.

Proof. Let $\mathbf{x}_i = P\{\tilde{M}_i > 0\}$. We bound the mathematical expectation of the random variable \tilde{M}_i :

$$E(\tilde{M}_i) = \int_{\tilde{M}_i > 0} \tilde{M}_i dP + \int_{\tilde{M}_i \leq 0} \tilde{M}_i dP \leq P\{\tilde{M}_i > 0\} = \mathbf{x}_i. \quad (50)$$

$$E(\tilde{M}_i) \geq -P\{\tilde{M}_i \leq 0\} = \mathbf{x}_i - 1. \quad (51)$$

Define a sequence of stock prices: $S_0 = 1/2$ and for $1 \leq i \leq 1$

$$S_i = \begin{cases} S_{i-1} - 2^{-(i+1)} & \text{if } \mathbf{x}_i > \frac{1}{2} \\ S_{i-1} + 2^{-(i+1)} & \text{otherwise.} \end{cases}$$

By definition $S_i > 0$ for all i .

Define the decision rule D :

$$D(\mathbf{x}_i) = \begin{cases} -1 & \text{if } \mathbf{x}_i > \frac{1}{2} \\ 1 & \text{otherwise.} \end{cases}$$

If $\mathbf{x}_i > \frac{1}{2}$ then $E(\tilde{M}_i) \geq -\frac{1}{2}$ by (51), $\Delta S_i = -2^{-(i+1)}$, and $D(\mathbf{x}_i) = -1$ by definition.

2. The internal randomizations is performed independently. Theorem 8 can be generalized to arbitrary sequence $\tilde{M}_1, \tilde{M}_2, \dots$ not necessary i.i.d.

If $\mathbf{x}_i \leq \frac{1}{2}$ then $E(\tilde{M}_i) \leq \frac{1}{2}$ by (50), $\Delta S_i = 2^{-(i+1)}$, and $D(\mathbf{x}_i) = 1$ by definition. We have

$$\begin{aligned} E\left(\sum_{i=1}^n \tilde{M}_i \Delta S_i\right) &= \sum_{i=1}^n E(\tilde{M}_i) \Delta S_i = \\ &= \sum_{\mathbf{x}_i > \frac{1}{2}}^n E(\tilde{M}_i) \Delta S_i + \sum_{\mathbf{x}_i \leq \frac{1}{2}}^n E(\tilde{M}_i) \Delta S_i \leq \frac{1}{2} \sum_{i=1}^n 2^{-(i+1)} = \frac{1}{4} \end{aligned} \quad (52)$$

for all n .

$$\sum_{i=1}^n D(\mathbf{x}_i) \Delta S_i = \sum_{\mathbf{x}_i > \frac{1}{2}}^n D(\mathbf{x}_i) \Delta S_i + \sum_{\mathbf{x}_i \leq \frac{1}{2}}^n D(\mathbf{x}_i) \Delta S_i = \sum_{i=1}^n 2^{-(i+1)} = \frac{1}{2}. \quad (53)$$

By the martingale law of large numbers (Azuma–Hoeffding inequality) with probability one

$$\frac{1}{n} \sum_{i=1}^n (\tilde{M}_i - E(\tilde{M}_i)) \rightarrow 0$$

as $n \rightarrow \infty$. From this (49) follows. Theorem is proved.

The discontinuous trading strategy D defined in Theorem 8 is unstable under small changes of the signal \mathbf{x}_i . In the next theorem, we show that if we randomly round the signal $\tilde{\mathbf{x}}_i$ then our universal trading strategy \tilde{M}_i (and \tilde{M}_i^1), performs at least as good as D .

Consider the protocol of trading game with two players and a side information $\mathbf{x}_i \in [0, 1]$ (see Fig 3).

We specify the information vector using by our universal strategy \tilde{M}_i to be $\bar{x}_i = (S_{i-1}, \mathbf{x}_i)$, where S_{i-1} is the past price of the stock and \mathbf{x}_i is the signal at step i . The universal trading strategy \tilde{M}_i uses the sequential method of randomization defined in Section 2 to perform a randomized forecast \tilde{p}_i and a randomized information vector $\tilde{x}_i = (\tilde{S}_{i-1}, \tilde{\mathbf{x}}_i)$.

The strategy of *Trader M* is the same as before:

$$\tilde{M}_i = \begin{cases} 1 & \text{if } \tilde{p}_i > \tilde{S}_{i-1}, \\ -1 & \text{otherwise,} \end{cases}$$

except that it uses a slightly different randomization.

Theorem 9 *An algorithm for computing forecasts and a sequential method of randomization can be constructed such that for any nontrivial decision rule D*

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{M}_i \Delta S_i - \frac{1}{n} \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i) \Delta S_i \right) \geq 0 \quad (54)$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

Moreover, for any $\epsilon > 0$ this trading strategy \tilde{M}_i can be tuned such that for any $\delta > 0$, with probability at least $1 - \delta$, for all nontrivial nonnegative decision rule $D \in \mathcal{F}$

$$\begin{aligned} \sum_{i=1}^n \tilde{M}_i \Delta S_i &\geq \|D\|_\infty^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i) \Delta S_i - \\ &- 5(m+1)en^{\frac{4}{5}+\epsilon} - (m+1)(e-1)\frac{4}{3}n^{\frac{3}{4}+\epsilon} - (m+1)\sqrt{\frac{n}{2} \ln \frac{2m}{\delta}} \end{aligned} \quad (55)$$

for all n , where m is the cardinality of the range of D .

Proof. For simplicity, we give the proof for the case of nonnegative decision rule and the randomized strategy M_i^1 . The case of arbitrary decision rule D and strategy \tilde{M}_i is considered similarly.

We apply Theorem 1 to zero kernel $R(\mathbf{x}, \mathbf{x}') = 0$ with $c_{\mathcal{F}} = 0$ and to the information vector $\bar{x}_i = (S_{i-1}, \mathbf{x}_i)$, $k = 2$.

Recall that $\epsilon > 0$ and $M = \lceil 1/\epsilon \rceil$. At any step i we compute the deterministic forecast p_i defined in Theorem 1 (Section 3) and its randomization to \tilde{p}_i using parameters $\Delta = \Delta_s = (s+M)^{-\frac{M}{4}}$ and $n_s = (s+M)^M$, where $n_s \leq i < n_{s+1}$.

The following upper bound is valid:

$$\begin{aligned} \left| \|D\|_\infty^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i) (\tilde{S}_{i-1} - S_{i-1}) \right| &\leq \sum_{t=0}^s (n_{t+1} - n_t) \Delta_t \leq \\ &\leq \frac{4}{3}(e-1)n_s^{\frac{3}{4}+\epsilon} \leq \frac{4}{3}(e-1)n^{\frac{3}{4}+\epsilon}. \end{aligned} \quad (56)$$

Let $D(\mathbf{x})$ be an arbitrary nontrivial nonnegative decision rule. Let \tilde{M}_i^1 be the randomized trading strategy defined in Section 4. We use abbreviations:

$$\nu_1(n) = (e-1)\frac{4}{3}n^{\frac{3}{4}+\epsilon}, \quad (57)$$

$$\nu_2(n) = 4e \left(\frac{3}{2} \right)^{\frac{2}{5}} n^{\frac{4}{5}+\epsilon} + \sqrt{\frac{n}{2} \ln \frac{2m}{\delta}}. \quad (58)$$

All sums below are for $i = 1, \dots, n$. By definition $0 \leq D(\tilde{\mathbf{x}}_i) \leq \|D\|_\infty$ for all $\mathbf{x}_i \in [0, 1]$.

Let d_1, \dots, d_m be all values of D . Define

$$\mathcal{S}_j = \{(p, y, \mathbf{x}) : 0 \leq p, y \leq 1, D(\mathbf{x}) = d_j\},$$

where $j = 1, \dots, m$; let $I_{\mathcal{S}_j}$ be the characteristic function of the set \mathcal{S}_j .

Let $\delta > 0$. Then, with probability $1 - \delta$, the following chain of equalities and inequalities is valid:

$$\begin{aligned} \sum_{i=1}^n \tilde{M}_i^1 (S_i - S_{i-1}) &= \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - S_{i-1}) = \\ &= \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - \tilde{p}_i) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{S}_{i-1} - S_{i-1}) \geq \end{aligned} \quad (59)$$

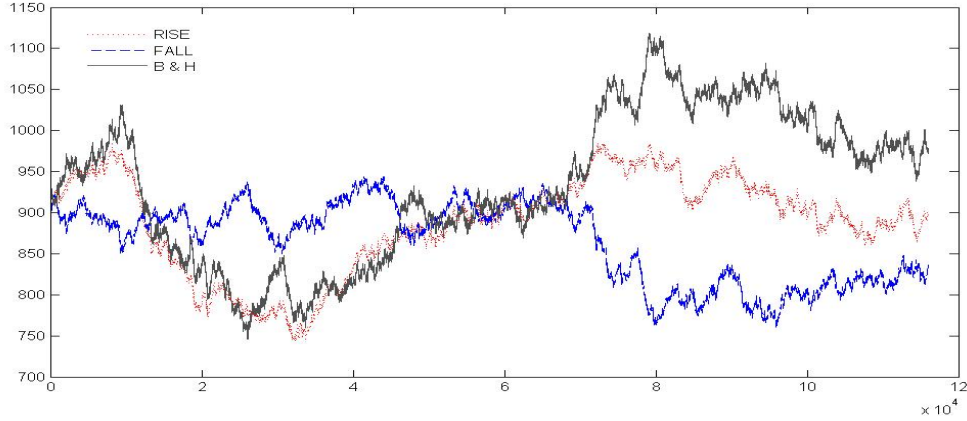


Figure 4: Evolution of capitals of three trading strategies for the period 26.03.10–25.03.11: Buy and Hold – solid line, UN dealing for a rise – dotted line, UN dealing for a fall – dashed line. One run of trading is performed with a simulated stock TEST (see Table 1)

$$\geq \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) \geq \quad (60)$$

$$\geq \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(\tilde{p}_i - \tilde{S}_{i-1}) - \nu_1(n) - \nu_2(n) = \quad (61)$$

$$\begin{aligned} &= \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(\tilde{p}_i - S_i) + \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(S_{i-1} - \tilde{S}_{i-1}) + \\ &\quad + \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(S_i - S_{i-1}) - \nu_1(n) - \nu_2(n) \geq \quad (62) \end{aligned}$$

$$\geq \|D\|_{\infty}^{-1} \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(S_i - S_{i-1}) - (1+m)\nu_1(n) - (1+m)\nu_2(n). \quad (63)$$

In change from (59) to (60) and in change from (62) to (63) we have used the inequality (56). In change from (62) to (63) we have used also Theorem 1, where $k = 2$, and, with probability $1 - \delta$,

$$\left| \sum_{i=1}^n D(\tilde{\mathbf{x}}_i)(S_i - \tilde{p}_i) \right| \leq \left| \sum_{j=1}^m d_j \sum_{i=1}^n I_{S_j}(\tilde{\mathbf{x}}_i)(S_i - \tilde{p}_i) \right| \leq m \|D\|_{\infty} \nu_2(n).$$

The inequality (54) follows from (55). Theorem 9 is proved.

Two symmetric solid lines gains of two zero sums strategies, dotted line expected gain of the algorithm PROT (without transaction costs), dashed line volume of the game

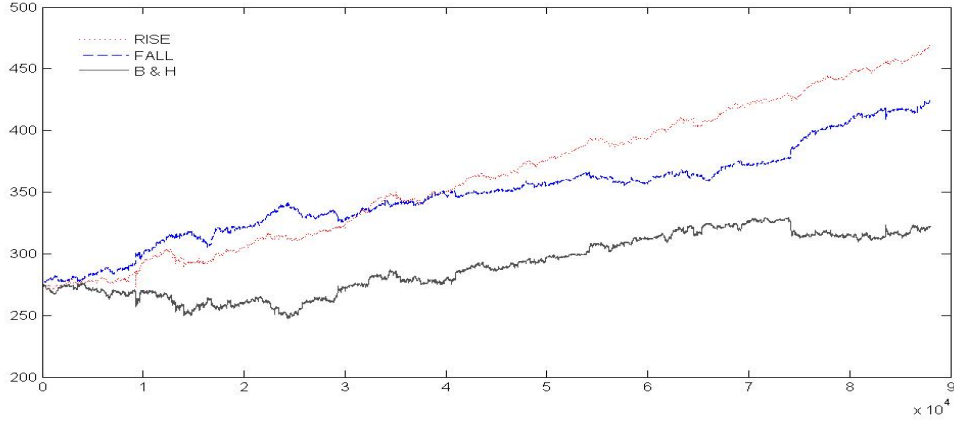


Figure 5: Evolution of capitals of three trading strategies for the period 26.03.10–25.03.11: Buy and Hold – solid line, UN dealing for a rise – dotted line, UN dealing for a fall – dashed line. One run of trading is performed with the stock KOCO (see also Table 1)

7. Numerical experiments

Computer technology. In the numerical experiments, we have used historical data in the form of per minute time series of prices of arbitrarily chosen stocks.

Two types of kernel functions were used as the smooth approximations of the combined kernel $K(p, x_n, p_i, \bar{x}_i) + R(\mathbf{x}_n, \mathbf{x}_i)$ from the sum (6): (i) $\mathcal{K}(p, p_n) = \cos((\pi(p - p_n)/2)$, (ii) $\mathcal{K}(p, p_n, x, x_i) = \exp(c(p - p_n) + c'(x - x_i))$, where c, c' are positive constants.

In any short-term trading algorithm, the time characteristics are crucial. The greatest time cost is associated with the calculation of sums (6) and finding the roots of this equation. The performed experiments show that the computation time for one point of the forecast increases linearly with increasing length of history. To provide one point of time, predicting within 1 - 3 seconds of CPU time, the length of the series was limited up to 5000 points. For series of length greater than 5000 points, “a chain” method of forecasting was used. Two processes working on overlapping intervals of time series are performed at the same time (see Fig 6).

Let L_{\max} be the chain length, and L_{shift} be the value of time shift, where $L_{\text{shift}} < L_{\max}$. In any process, the first L_{shift} time-points are used only for scaling prices and preliminary learning of the forecasting algorithm. The trading is not performed at first L_{shift} time-points of the series.

When a regular process terminates we switch to the time-point $L_{\text{shift}} + 1$ of the next process. The results of parallel computing are accumulated into a single overall forecasting series. We get $L_{\max} = 5000$ and $L_{\text{shift}} = 2000$.

The prices of a stock are scaled such that $S_i \in [0, 1]$ for all i . The scaling is performed for time series of each process separately. The first L_{shift} time points of any process are

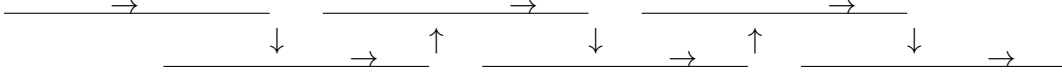


Figure 6: Scheme of parallel computations

used for computing a scaling constant. Prices are scaled as follows:

$$S_i = \frac{\hat{S}_i}{c \max_{1 \leq j \leq L_{\text{shift}}} \hat{S}_j},$$

where $1 \leq i \leq L_{\text{max}}$ and \hat{S}_i are real prices of the stock. We get $c = 14$.

The forecasting algorithm is performed for the scaled prices S_i , where $L_{\text{shift}} + 1 \leq i \leq L_{\text{max}}$.

We implement this computer technology for two forecasting algorithms: the universal strategy constructed in Section 3 (UN-model) and Autoregressive Moving Average algorithm (ARMA-model) (cf. Peng and Aston 2011).³

Results of numerical experiments. In the numerical experiments, we have used historical data in form of per minute time series of prices of arbitrarily chosen 17 stocks (11 US stocks, and 6 Russian stocks) and of one simulated stock TEST. Data has been downloaded from FINAM site: www.finam.ru. Number of trading points in each game is $N=88000-116000$ min. (From March 26 2010 to March 25 2011).

The artificial stock TEST is simulated as $S_i = S_{i-1} + \xi_i$, $i = 1, 2, \dots, N$, where ξ_i is the Gaussian random variable with mean 0 and a variance equal to the variance of the scaled GAZP stock.

We implement the trading strategy defined in Section 4.

Two series of numerical experiments were performed.

In the first series, we use the trading strategy \tilde{M}_i studied in Theorem 5. At each step, starting from initial capital $\mathcal{K}_0^R = \mathcal{K}_0^F = \mathcal{K}_0 = K S_0$, where S_0 is the price of a stock at the first time point, this strategy performs dealing for a rise or for a fall with K shares of the stock. We take $K = 5$ in our experiments. In case of dealing for a rise, the capital changes at any step i as $\mathcal{K}_i^R = \mathcal{K}_{i-1}^R + K(S_i - S_{i-1})$ if $\tilde{p}_i > \tilde{S}_{i-1}$ and $\mathcal{K}_i^R = \mathcal{K}_{i-1}^R$ otherwise. In case of dealing for a fall $\mathcal{K}_i^F = \mathcal{K}_{i-1}^F - K(S_i - S_{i-1})$ if $\tilde{p}_i \leq \tilde{S}_{i-1}$ and $\mathcal{K}_i^F = \mathcal{K}_{i-1}^F$ otherwise, where $i = 1, 2, \dots, N$.

Results of numerical experiments are shown in Table 1. In the first column, stocks ticker symbols are shown. The second column contains the profit of Buy-and-Hold trading strategy. By this strategy, we buy a holding of shares using capital \mathcal{K}_0 and sell them for \mathcal{K}_N at the end of the trading period.

3. See also the State Space Models Toolbox for MATLAB:
<http://sourceforge.net/projects/ssmodels/>.

Table 1: Universal trading

TICKER	BUY& HOLD PROFIT %	UN FOR A RISE PROFIT %	UN FOR A FALL PROFIT %	ARMA FOR A RISE PROFIT %	ARMA FOR A FALL PROFIT %
TEST	6.85	-1.39	-8.19	9.88	3.08
AT-T	7.71	145.21	137.51	53.74	46.038
CTGR	15.04	1711.47	1696.78	1534.72	1620.03
KOCO	16.55	69.84	53.32	31.35	14.83
GOOG	10.25	115.80	105.57	43.01	32.78
INBM	24.28	83.59	59.30	53.45	29.16
INTL	4.29	118.91	114.71	58.57	54.37
MSD	10.71	56.40	45.69	33.90	23.19
US1.AMT	22.01	28.37	6.40	28.97	7.00
US1.IP	2.40	30.12	27.75	-7.88	-10.24
US2.BRCM	25.30	62.53	37.19	49.20	23.86
US2.FSLR	40.15	159.11	118.81	13.73	-26.56
SIBN	-6.54	747.80	754.27	448.12	444.89
GAZP	22.75	100.01	77.26	37.89	15.14
LKOH	19.39	269.07	249.67	136.2 9	116.89
MTSI	-1.61	698.08	699.61	434.84	436.37
ROSN	9.69	197.03	187.26	93.99	84.22
SBER	14.21	112.05	97.98	32.62	18.55

In the 3th and 4th columns, results of one run of trading based on the universal randomized forecasting strategy (UN) are shown. In the 3th column, a relative return, percentagewise, to the initial capital $\frac{\mathcal{K}_N - \mathcal{K}_0}{\mathcal{K}_0} 100\%$ is shown for dealing for a rise, in the 4th column, the same relative return is shown for dealing for a fall, In the 5th and 6th columns, the same results are shown for trading using ARMA forecasts.

It was found that $\mathcal{K}_i > 0$ for $i = 1, 2, \dots, N$, i.e., we never incur debt in our experiments (with an exception of TEST stock).

Results presented in Table 2 show that trading based on UN model of forecasting performs at least as good as the trading based on ARMA forecasting model and essentially outperforms it for some stocks.

The second series of experiments is closer to a real short-term trading. The trading strategy has a defence guarantee. Starting with the same initial capital $\mathcal{K}_0 = K S_0$, where S_0 is the initial price of a stock and $K = 5$, we perform dealing for a rise using “a defensive” trading strategy. At any step i , our working capital is $\mathcal{L}_{i-1} = \min\{\mathcal{K}_0, \mathcal{K}_{i-1}\}$. Using this capital, we buy $M_i = \mathcal{L}_{i-1}/S_{i-1}$ shares of the stock at the beginning of any step i , if $\mathcal{L}_{i-1} > 0$, and stop trading otherwise: $M_i = 0$. We update the cumulative capital at the end of each step: $\mathcal{K}_i = \mathcal{K}_{i-1} + M_i(S_i - S_{i-1})$. Thereby, we can set aside the extra income.

Results of second series of numerical experiments are shown in Table 2. In the first column, stocks ticker symbols are shown. The second column contains the relative return of Buy-and-Hold trading strategy. In the next pair of columns marked “UN”, relative returns of one run of randomized trading, percentagewise, for the initial capital are presented for the case with no transaction costs and for the case where transaction cost at the rate 0.01%

Table 2: Defensive trading

TICKER	BUY& HOLD %	UN PROFIT %	UN PROFIT -0.01%	ARMA PROFIT %	ARMA PROFIT -0.01%	UN IN	ARMA IN	UN D	ARMA D
TEST	6.85	3.58	-80.93	3.58	-80.90	0.232	0.163	1.453	1.890
AT-T	7.71	69.01	-79.19	29.86	-79.19	0.218	0.205	1.611	1.576
CTGR	15.04	1030.12	658.13	937.46	540.18	0.238	0.253	1.654	1.479
KOCO	16.55	36.47	-78.62	15.69	-78.55	0.216	0.198	1.604	1.502
GOOG	10.25	46.54	-80.57	3.53	-82.68	0.231	0.211	1.462	1.474
INBM	24.28	54.79	-78.53	34.66	-78.10	0.219	0.187	1.514	1.517
INTL	4.29	43.06	-76.60	5.63	-76.28	0.220	0.179	1.630	1.585
MCD	10.71	34.22	-78.56	19.21	-78.41	0.222	0.190	1.571	1.876
AMT	22.01	16.47	-77.01	24.04	-77.09	0.212	0.183	1.654	1.758
IP	2.40	4.45	-82.78	-14.79	-81.06	0.213	0.181	1.657	1.760
BRCM	25.30	11.40	-80.47	23.98	-76.10	0.216	0.172	1.585	1.876
FLSR	40.15	21.02	-80.04	-27.50	-80.03	0.227	0.196	1.499	1.506
SIBN	-6.54	600.62	249.87	287.48	-58.55	0.169	0.179	2.460	2.292
GAZP	22.75	51.29	-82.04	4.34	-82.16	0.224	0.210	1.539	1.526
LKOH	19.39	149.03	-79.91	46.44	-80.62	0.230	0.244	1.527	1.501
MTSI	-1.61	482.83	79.23	275.13	-69.36	0.188	0.195	2.174	1.959
ROSN	9.69	101.15	-83.14	-0.53	-83.54	0.228	0.240	1.549	1.499
SBER	14.21	51.56	-82.52	-14.47	-82.73	0.225	0.196	1.559	1.674

is subtracted. We compute the forecast of a future stock price by the method of calibration and defensive forecasting (UN) presented in Theorem 1.

The next two columns marked by “ARMA” are similar, with the exception that the ARMA forecasting model is used for computing forecasts. The frequencies of market entry steps i , where $\tilde{p}_i > \tilde{S}_{i-1}$, are given in the next two columns marked “In” (for UN and ARMA). We sell all shares of a stock at step i in case $\tilde{p}_i \leq \tilde{S}_{i-1}$. The average time spent in the market is shown in the rest two columns marked “D” (for UN and ARMA).

8. Conclusion

Asymptotic calibration is an area of intensive research where several algorithms for computing well-calibrated forecasts have been developed. Several applications of well-calibrated forecasting have been proposed (convergence to correlated equilibrium, recovering unknown functional dependencies, predictions with expert advice). We present a new application of the calibration method.

We show that the universal trading strategy can be constructed using the well-calibrated forecasts. We prove that this strategy performs at least as good as any stationary trading strategy presented by a rule from any RKHS with regret $O(n^{\frac{3}{4}})$. Using the universal kernel, we prove that this strategy performs at least as good as any stationary continuous trading strategy.

The obvious drawback of a universal strategy is that it uses the high frequency trading, which prevents it from practical applications in the presence of transaction costs.

To construct the universal trading strategy, we generalize Kakade and Foster's algorithm and combine it with Vovk's DF-model for arbitrary RKHS. Using Vovk (2006) theory of defensive forecasting in Banach spaces, these results can be generalized to these spaces.

Unlike the statistical theory, no stochastic assumptions are made about the stock prices.

Numerical experiments show a positive return for all chosen stocks, and for some of them we receive a positive return even when transaction costs are subtracted. Results of this type can be useful for technical analysis in finance.

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